Cuspidal Point on the Primer Locus

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The necessary conditions for an optimal impulsive, time-free, orbit transfer are conveniently expressed in terms of the primer vector, the adjoint vector associated with the velocity vector. For a successful computation and tabulation of all optimal arcs in three-dimensional transfer, it is necessary to investigate the limiting cases of different families of optimal arcs. The family of Forward-Forward (or Rearward-Rearward) transfers tends to the Lawden's singular arc when the transfer angle tends to zero. All the elements of the singular arc are expressed explicitly in terms of the azimuth and the elevation of the optimum thrust direction and a directional parameter. The family of Forward-Rearward transfers tends to the symmetric arc. All the elements of the symmetric arc can be expressed in terms of the eccentricity of the transfer orbit and the azimuth of the first thrust direction. Another remarkable limiting case is the case where a cusp appears on the primer locus at a point of application of an impulse. For the planar transfer this case is a limiting case in the sense that the primer locus changes from a two-loop curve to a one-loop curve. The point where the cusp appears is very close to the point of highest elevation angle for the optimum thrust. Explicit formula for the computation of the optimum cuspidal arcs are given.

Nomenclature

a,b,c,d,f = constants of integration in the equations for the primer vector
A,B,C = coefficients, Eqs. (12) and (50)
A_1,A_2,A_3 = coefficients, Eq. (15)
e = eccentricity of the transfer orbit
J_1,J_2 = parameters, defined in Eq. (5)
k = $\cos v$
K_1,K_2 = parameters, defined in Eq. (5)
m = $\cos\Delta$
n = $\cos 2\phi_1$
$\mathbf{p}$ = primer vector
r = $1 + e \cos v$
S,T,W = radial, circumferential, and normal components of the primer vector
t = time
U = $(T_1 - T_2)/\theta$, when $T_1 \to T_2$, $\theta \to 0$, Eq. (31)
v = true anomaly along the transfer orbit
x = $e \cos v$
y = $e \sin v$
α = $\tan(\phi_2 - \phi_1)/2$
Δ = $v_2 - v_1$
θ = $\tan(v_2 - v_1)/2$
λ,ω = constants of integrations, defined in Eq. (19)
ρ = radius of circle, locus of $\mathbf{p}_1$, Eq. (19)
ϕ = elevation of the optimum thrust
ψ = azimuth of the optimum thrust

I. Introduction

THE necessary conditions for an optimal impulsive orbit transfer are conveniently expressed in terms of the primer vector, the adjoint vector associated with the velocity vector.[1] Recent developments in the use of the primer vector[2–7] have brought new interest in that powerful concept, first formulated by Lawden. In terms of the primer vector, denoted in this paper by $\mathbf{p}$, we have the following necessary conditions for an optimal transfer[1] 1) the primer vector, as an ajoint vector, must be continuous, together with its first derivative, everywhere; 2) the magnitude of the primer vector must not exceed unity during the transfer ($p \leq 1$); and 3) the impulsive thrust must be applied in the direction of the primer vector at the times for which $p = 1$.

Hence, the analysis of the optimum orbit transfer reduces to the study of the vector function $\mathbf{p} = \mathbf{p}(t)$ which gives the optimal direction for the impulse, and the scalar function $f(t) = 1 - p^2$, which gives the time t_i of the application of the impulse whenever $f(t_i) = 0$. When the time varies, the terminus of the vector $\mathbf{p}$ describes a curve, called the primer locus. Hence, if t_i, with $i = 1,2, \ldots n$, are the times of application of an impulse, the space curve, defined by the function $\mathbf{p} = \mathbf{p}(t)$, and the unit sphere have a contact point of at least order zero (intersection) at t_1 and t_n, and a contact point of at least order unity (tangency) at $t_2, \ldots, t_{n-1}$ (Fig. 1).

This assumes that all the contact points are regular points on the primer locus. In this paper we examine the cases where one of the contact points is irregular, that is when the contact is of higher order (Lawden's singular case) or when the contact point is a cuspidal point on the primer locus. We consider the case of free time for the transfer, and in this case, along an intermediary transfer orbit, the primer locus is a closed curve. It will be shown that, for the Lawden's noncoplanar singular arc, all the extremum elements can be expressed in terms of a family of three parameters. For the cuspidal case, the optimum elements are also expressed in terms of a family of three parameters. All the expressions are given in explicit forms.

II. Primer Vector along a Ballistic Arc

Consider an arc I_1I_2 of a transfer orbit connecting two orbits O_1 and O_2. The rocket entered the coasting arc at the point

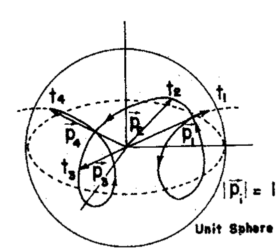

Fig. 1 Example of primer locus for fixed-time four-impulse, space transfer.

Presented as Paper 70-1037 at the AAS/AIAA Astrodynamics Conference, Santa Barbara, Calif., August 19–21, 1970; submitted September 18, 1970; revision received June 23, 1971. Work supported by NASA Contract NASr 54(06) and Air Force Grant AFOSR 71-2129.

Index category: Navigation, Control, and Guidance Theory.

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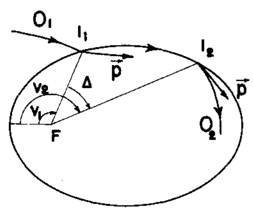

Fig. 2 The coasting arc.

I_1 and left it at the point I_2 (Fig. 2). We shall use subscript 1 to denote the elements corresponding to the first impulse, and subscript 2 for the second impulse. The points I_1 and I_2 are called the switching points and the pair of points I_1 and I_2 forms a switching. The positions of the switching points are defined by their true anomalies v_1 and v_2 measured along the ballistic arc. The angle $\Delta = v_2 - v_1$ is the transfer angle. To relate the optimal elements at a switching, it suffices to express the variation of the primer along the ballistic arc in such a way that the primer locus is tangent to the unit sphere at the switching points.

The solution for the primer, along a ballistic arc, in an inverse square force field, has been obtained by Lawden.[1] Using a system of rotating axes $MSTW$ (Fig. 3) such that the origin is at the position of the rocket, considered as a mass point, the S axis along the position vector, positive outwards, the T axis along the circumferential direction, positive toward the direction of motion, and the W axis completing a right-handed system, we have for the components of the primer vector $\mathbf{p}(v)$, considered as function of the true anomaly v, in the time-free problem

$$\begin{aligned} S &= a\cos v + be\sin v \\ T &= -a\sin v + b(1 + e\cos v) + (c - a\sin v)/(1 + e\cos v) \qquad (1) \\ W &= (d\cos v + f\sin v)/(1 + e\cos v) \end{aligned}$$

where a, b, c, d, and f are arbitrary constants of integration, and e the eccentricity of the coasting, or null-thrust, Keplerian orbit along which the rocket is traveling. Equations (1) are parametric equations of the primer locus in the rotating system. In this system, the optimum direction of the thrust, whenever it is active, is defined by the projection S, T, W of the primer on the axes, or in an alternate way, by its azimuth ψ, and elevation ϕ. Therefore, at the switching points, we have

$$\begin{aligned} &S_i = \sin\phi_i,\ T_i = \cos\phi_i\cos\psi_i,\ W_i = \cos\phi_i\sin\psi_i \\ &S_i^2 + T_i^2 + W_i^2 = 1 \qquad i = 1,2 \end{aligned} \qquad (2)$$

By writing the equations of the primer at the switching points

$$\begin{aligned} S_i &= a\cos v_i + be\sin v_i \\ T_i &= -a\sin v_i + b(1 + e\cos v_i) + \\ &\qquad (c - a\sin v_i)/(1 + e\cos v_i) \qquad (3) \\ W_i &= (d\cos v_i + f\sin v_i)/(1 + e\cos v_i) \end{aligned}$$

where $i = 1,2$. By elimination of the five constants of integration among these six relations we have

$$(1 + r_1)J_1 - r_1T_1 = (1 + r_2)J_2 - r_2T_2 \qquad (4)$$

where in the following we shall use the notation

$$x = e\cos v,\ y = e\sin v,\ \Delta = v_2 - v_1,\ r = 1 + e\cos v \qquad (5a)$$

$$J_1 = (S_2 - S_1\cos\Delta)/\sin\Delta,\ J_2 = (S_2\cos\Delta - S_1)/\sin\Delta \qquad (5b)$$

$$K_1 = (r_2W_2 - r_1W_1\cos\Delta)/\sin\Delta, \qquad (5c)$$

$$K_2 = (r_2W_2\cos\Delta - r_1W_1)/\sin\Delta \qquad (5d)$$

By expressing that the primer locus is tangent to the unit sphere, centered at M, at the switching points, that is $SdS + TdT + WdW = 0$ at these points, we have

$$[y_1T_1 - r_1S_1](T_1 - J_1) - S_1T_1 + y_1W_1^2 + K_1W_1 = 0 \qquad (6)$$

and

$$[y_2T_2 - r_2S_2](T_2 - J_2) - S_2T_2 + y_2W_2^2 + K_2W_2 = 0 \qquad (7)$$

In summary, among the nine elements v_i, S_i, T_i, W_i, and e, relating a switching, we have Eqs. (4, 6, and 7), called the switching equations, and the two constraints

$$S_i^2 + T_i^2 + W_i^2 = 1 \qquad (8)$$

Hence, the switching equations contain only seven independent parameters, say the true anomalies v_i of the switching points, the optimal thrust angles ϕ_i,ψ_i at these points, and the eccentricity e of the ballistic arc. The system of switching equations can be solved for any three of the parameters in terms of the remaining four. An alternate derivation of the switching equations was given in Ref. 7. It is shown by Marchal in the same reference that, if the thrust directions at the switching points are known, that is if the four angles ϕ_i,ψ_i are given, the solution to switching equations can be obtained by solving the cubic equation in θ

$$\begin{aligned} &2(S_2 + S_1)^2(T_2 - T_1)\theta^3 + (S_2 + S_1)[-(S_2 + S_1)^2 + \\ &\quad 3(T_2 - T_1)^2 + (W_2 + W_1)^2 + 2(W_2 - W_1)^2]\theta^2 + \\ &\quad \{4(S_2^2 - S_1^2)(T_2 + T_1) + (T_2 - T_1)[(T_2 + T_1)^2 + \\ &\quad (T_2 - T_1)^2 + (W_2 + W_1)^2 + (W_2 - W_1)^2]\}\theta + \\ &\quad (S_2 + S_1)[-3(S_2 - S_1)^2 + (T_2 - T_1)^2 + \\ &\quad (W_2 - W_1)^2] = 0 \qquad (9) \end{aligned}$$

where

$$\theta = \tan\Delta/2 \qquad (10)$$

We shall refer to this remarkable equation as Marchal's equation. The equation has the advantage of retaining a symmetric form for the optimal thrust angles. For this paper we shall use Marchal's equation in the form

$$A\cos\psi_2 + B\sin\psi_2 + C = 0 \qquad (11)$$

where

$$\begin{aligned} A &= \cos\phi_2[-(1 + 3\theta^2)(\sin\phi_2 + \sin\phi_1)\cos\phi_1\cos\psi_1 + \\ &\quad \theta(2 + \sin^2\phi_2 - 3\sin^2\phi_1) + \theta^3(\sin\phi_2 + \sin\phi_1)^2] \\ B &= -(1 + \theta^2)(\sin\phi_2 + \sin\phi_1)\cos\phi_1\cos\phi_2\sin\psi_1 \\ C &= -\theta[(2 + \sin^2\phi_1 - 3\sin^2\phi_2) + \theta^2(\sin\phi_2 + \\ &\quad \sin\phi_1)^2]\cos\phi_1\cos\psi_1 + (\sin\phi_2 + \sin\phi_1)[(1 - \\ &\quad 2\sin^2\phi_1 - 2\sin^2\phi_2 + 3\sin\phi_1\sin\phi_2) + \\ &\quad \theta^2(3 - 2\sin^2\phi_1 - 2\sin^2\phi_2 - \sin\phi_1\sin\phi_2)] \end{aligned} \qquad (12)$$

In this form, we can see that if the four angles Δ, ϕ_1, ϕ_2, and ψ_1 (or ψ_2) are given, then the angle ψ_2 (or ψ_1) can be evaluated in closed form, and subsequently the remaining elements v_1, v_2, and e can be calculated.

Since the magnitude of the primer vector is equal to unity at a switching point, the scalar function $f(v) = 1 - p^2$, called the switching function, gives the position v_i of the impulse

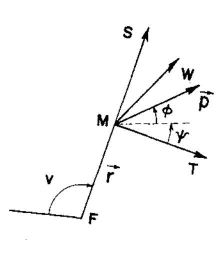

Fig. 3 The rotating axes.

whenever $f(v_i) = 0$. As given in Ref. 7, we have

$$f(v) = [1 - \cos(v - v_1)][1 - \cos(v - v_2)]F(v)/r^2 \quad (13)$$

where

$$F(v) = A_1 \cos v + A_2 \sin v + A_3 \quad (14)$$

and

$$A_1 = 2e(S_1S_2 - J_1J_2) \quad (15a)$$

$$A_2 = 2e(S_1J_2 + S_2J_1) \quad (15b)$$

$$A_3 = (1 + \theta^2)[(T_1 - 2J_1)(2J_1 - x_1T_1) - x_1W_1^2] + A_1 \cos v_1 + A_2 \sin v_1 \quad (15c)$$

$$= (1 + \theta^2)[(T_2 - 2J_2)(2J_2 - x_2T_2) - x_2W_2^2] + A_1 \cos v_2 + A_2 \sin v_2 \quad (15d)$$

By property 2, mentioned in Sec. I, the function $F(v)$ is nonnegative along the transfer orbit. Also by expression (13) for $f(v)$, along the transfer orbit, the primer locus touches the unit sphere, in general, at the two points v_1, and v_2 corresponding to the switching points. In the special case where $F(v) = 0$ for a value v_3 in the closed interval $[0,2\pi]$, the primer locus has three contact points with the unit sphere, and the plot of the switching function has the form as shown in Fig. 4.

The switching function vanishes three times along the transfer orbit and this will require[7]

$$A_1^2 + A_2^2 = A_3^2 \quad (16)$$

This case corresponds to a three-impulse transfer with the second impulse being infinitesimal so that the equation of the primer remains unchanged throughout the interval from v_1 to v_3.

The equations derived in this section are sufficient for a computation and cataloguing of all three-dimensional optimal arcs connecting impulsive switchings. A successful completing of such cataloguing for the coplanar case has been carried out in Ref. 2. For the planar case, the family of optimal arcs depends on two parameters. It has been pointed out in Ref. 6 that, although it is possible to construct explicitly, in closed form, an example of two-impulse optimum transfer, the computation of the optimal arc, starting with a specified eccentricity of the transfer orbit, requires the solving of an octic equation. For the noncoplanar case, the family of optimal arcs depends on four parameters. The tabulation of the optimum elements is still feasible and, evidently, this extensive enterprise would be greatly simplified if a rational approach is taken. Optimal arcs can be classified into two families. The first family of optimal arcs consists of ballistic arcs connecting a Forward-Forward (or Rearward-Rearward) switching. A Forward impulse is an impulse in support of the motion and it is characterized, in three-dimensional transfer, by a positive T component of the primer vector. Conversely, a Rearward impulse is an impulse which opposes the motion, and it corresponds to a negative value for the component T. The second family of optimal arcs consists of ballistic arcs connecting a Forward-Rearward switching. It is shown in Ref. 2, for coplanar transfer, that the first family of optimal arcs tends to the Lawden's point of self-switching and the second family of optimal arcs tends to a symmetric switching. They are limiting cases and all optimal arcs can be scanned by starting from the apsidal switching and progressively going to the limiting switchings.[6]

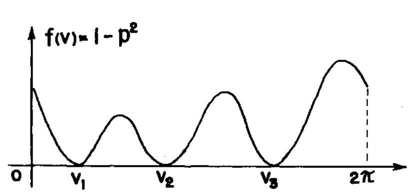

Fig. 4 The switching function.

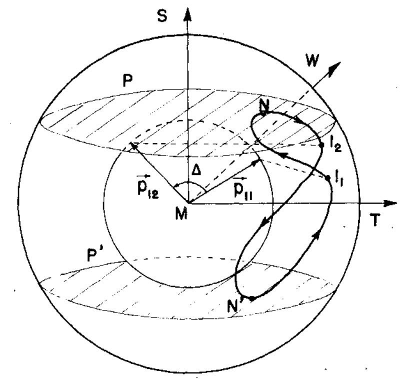

Fig. 5 Primer locus in free-time transfer.

The limiting switchings correspond to characteristic points on the primer locus. It is the purpose of this paper to investigate these cases. The existence of the different characteristic points on a primer locus can be best visualized by a geometric representation of the primer vector as introduced in Ref. 6. From the parametric representation (1) of the primer vector, we can see that it is the sum of three vectors.

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \quad (17)$$

with the components along the STW axes

$$\mathbf{p}_1 = [\rho \sin(v + \omega), \rho \cos(v + \omega), 0] \quad (18a)$$

$$\mathbf{p}_2 = \{0, [\lambda + \rho \cos(v + \omega)]/r, 0\} \quad (18b)$$

$$\mathbf{p}_3 = [0, 0, (d \cos v + f \sin v)/r] \quad (18c)$$

where

$$\rho = (a^2 + b^2e^2)^{1/2}, \omega = \tan^{-1}(a/be), \lambda = b + c \quad (19)$$

For each given switching configuration, when v varies from 0 to 2π, the terminus of the primer vector $\mathbf{p}$ describes the primer locus in such a way that its vector components $\mathbf{p}_1$ is always in the ST plane, describing a circle centered at M, having a radius ρ, the vector $\mathbf{p}_2$ always along the T axis, and the vector $\mathbf{p}_3$ always along the W axis. The primer locus is tangent to the unit sphere at the points I_1 and I_2 corresponding to the switching points (Fig. 5). We also notice that, if $\mathbf{p}_{11}$ and $\mathbf{p}_{12}$ are, respectively, the positions of the vector $\mathbf{p}_1$ at these switching points, the angle Δ between the positions is precisely the transfer angle. Furthermore, the primer locus is tangent to two planes P and P', parallel to the TW plane, at the altitude $\pm\rho$, at the points N and N', respectively. These points correspond to the positions of the vector $\mathbf{p}_1$ along the S axis. The following limiting cases will be considered; 1) the points I_1 and I_2 are along the unit circle in the TW plane (Apsidal switching), 2) the primer locus is symmetric with respect to the S-axis (Symmetric switching), 3) the points I_1 and I_2 are coincident (Lawden's singular case), and 4) the points I_2 and N are coincident (Cuspidal case).

It is shown in Ref. 6 that, for the planar case, the case 4 is a limiting case where, for a Forward-Forward (or Rearward-Rearward) type transfer, the form of the primer locus changes from a two-loop to a one-loop curve. In Fig. 5, it should be noticed that, between switching points, the useful part of the primer locus is limited between the points I_1 and I_2 (or I_1 and I_3 for the case of three contact points) although it is shown for the whole range of the true anomaly v from 0 to 2π along the transfer orbit.

III. Apsidal Switching

This limiting case is simple. It corresponds to the case of transfer between coaxial orbits. For a Forward-Forward

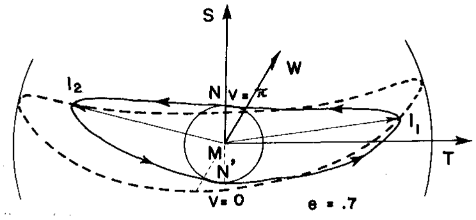

Fig. 6 Primer locus in a symmetric switching.

switching, we have

$$v_1 = 0,\ v_2 = \pi,\ \phi_1 = 0,\ \phi_2 = 0$$
$$\sin\psi_2 = -[(1+e)/(1-e)]\sin\psi_1 \tag{20}$$

For a Forward-Rearward switching, we have

$$v_1 = 0,\ v_2 = \pi,\ \phi_1 = 0,\ \phi_2 = \pi$$
$$\sin\psi_2 = [(1+e)/(1-e)]\sin\psi_1 \tag{21}$$

In both cases, all the optimal arcs can be obtained by using ψ_1 as parameter with

$$-\sin^{-1}[(1-e)/(1+e)] \leqq \psi_1 \leqq \sin^{-1}[(1-e)/(1+e)] \tag{22}$$

IV. Symmetric Switching

The symmetric case is obtained by writing that the primer locus is symmetric with respect to the S axis. The points N and N' are on the S axis with N' corresponding to $v = 0$ and N for $v = \pi$ (Fig. 6). We have

$$v_2 = -v_1,\ S_2 = S_1,\ T_2 = -T_1,\ W_2 = -W_1 \tag{23a}$$

$$\tan\phi_1 = -[x_1r_1/y_1(2+x_1)]\cos\psi_1 \tag{23b}$$

$$\tan\phi_1 = \frac{[y_1^2 + (2+x_1)(e^2+x_1)\tan^2\psi_1]\cos\psi_1}{y_1(3+2x_1)} \tag{23c}$$

Therefore

$$e^2(1+\tan^2\psi_1)k^3 + e[3+(4+e^2)\tan^2\psi_1]k^2 + [(3+e^2)+4(1+e^2)\tan^2\psi_1]k + 2e(1+2\tan^2\psi_1) = 0 \tag{24}$$
$$k = \cos v_1$$

If e and ψ_1 are given, the positions of the symmetric switching points are obtained by solving the cubic Eq. (24), and the other parameters can be obtained explicitly from Eq. (23). For planar transfer, $\psi_1 = 0$ and we have the well-known equation

$$e^2k^3 + 3ek^2 + (3+e^2)k + 2e = 0 \tag{25}$$

When ψ_1 varies from 0 to $\pm\pi/2$, v_1 varies from the root of this equation, say $\cos v_{11}$, to the limiting value $\cos v_{12} = -e$. The locus of the symmetric impulses I_1 and I_2 on the unit sphere, for a specified value of e, is plotted in dotted line in Fig. 6. The elevation angle ϕ_1 of the optimum thrust direction is decreasing from the maximum value corresponding to $\psi_1 = 0$ to the value zero when $\psi_1 = \pm\pi/2$. The radius ρ of the inner circle, locus of $\mathbf{p}_1$, also decreases from $-\sin\phi_1/k$ to zero. When $\psi_1 = 0$, the elevation angle is

$$\tan 2\phi_1 = -2\sin 2v_1/(3-\cos 2v_1) \tag{26}$$

Hence, the maximum elevation angle for symmetric switching is obtained when (Fig. 7)

$$\psi_1 = \psi_2 = 0,\ \cos 2v_1 = \tfrac{1}{3},\ v_1 = 144°44' \tag{27}$$

and this corresponds to

$$\tan\phi_1 = (3)^{1/2} - (2)^{1/2},\ \phi_1 = 17°38' \tag{28}$$

and an eccentricity

$$e = 3[(6)^{1/2} - 1]/5 = 0.86969 \tag{29}$$

The symmetric switching occurs in a two-impulse transfer between orbits symmetric with respect to a straight line, a special case being the planar rotation of orbit discussed in Ref. 8. Like in the planar transfer, it is the limiting case of the Forward-Rearward family of optimal arcs. Figure 7 plots the limits of the true anomaly v_1, and the elevation angle ϕ_1 when ψ_1 varies from 0 to $\pi/2$ for different values of the eccentricity. There is another case of symmetric switching, namely the case where the primer locus is symmetric with respect to the SW plane. This case is not discussed in this paper.

V. Lawden's Singular Case

This case occurs when the two switching points I_1 and I_2 are coincident and the contact point between the primer locus and the unit sphere is of order two. The transfer angle Δ tends to zero and, subsequently, we have

$$v_1 \to v_2,\ S_1 \to S_2,\ T_1 \to T_2,\ W_1 \to W_2,\ J_1 \to J_2 \tag{30}$$

Although, to the limit, the governing equations derived in Sect. II are still valid, they must be handled with care since the singularity makes the equations trivial. An excellent discussion of the primer locus in the planar singular case can be found in Ref. 2. In this paper, we shall give explicit expressions of the elements in the noncoplanar case when $W_1 = W_2$ identically.

At the limiting point, as defined by Eq. (30), let

$$J = \lim J_1 = \lim J_2,\ U = \lim(T_1 - T_2)/\theta \tag{31}$$

The first clue to the computation of the singular arc was given in Refs. 7 and 9. We have[7]

$$T_1 - 2J = y_1(1 - 3S_1^2)/2r_1S_1 \tag{32}$$

Since J is not known for the moment, we need two more equations to calculate x_1 and y_1 in terms of S_1 and T_1. Using the identities

$$x_2 = x_1\cos\Delta - y_1\sin\Delta,\ y_2 = x_1\sin\Delta + y_1\cos\Delta \tag{33}$$

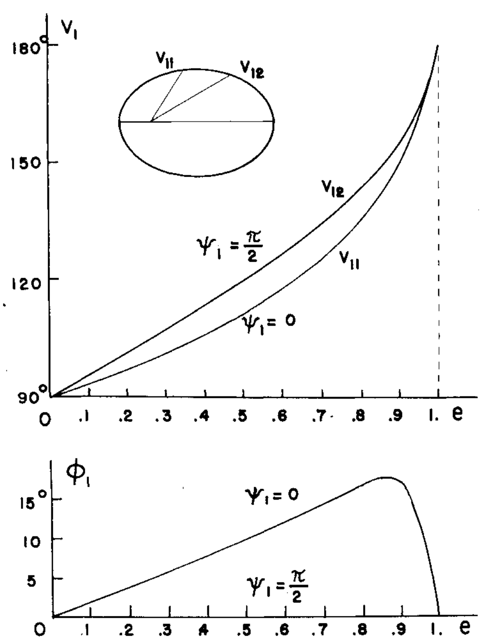

Fig. 7 Variations of the optimum position and thrust elevation angle for symmetric transfer.

we can write Eq. (6), with $\theta = \tan\Delta/2$, $W_1 = W_2$,

$$[y_1T_1 - r_1S_1](T_1 - J_1) - S_1T_1 + W_1^2\theta = 0$$

When $\theta \to 0$, we have the second equation for singular arc

$$T_1 - J = S_1T_1/(y_1T_1 - r_1S_1) \qquad (34)$$

Next, when $W_1 = W_2$, Marchal's equation, Eq. (9), is reduced to

$$(T_2 - T_1)^2 + 4W_1^2 + 3(J_1 + J_2)(T_1 + T_2) + 2(T_2 - T_1)(S_1 + S_2)\theta = (S_1 + S_2)^2 - (S_1 + S_2)(T_2 - T_1)\theta^{-1}$$

To the limit, we have

$$2W_1^2 + 6T_1J = 2S^2 + SU \qquad (35)$$

Using the identity

$$J_2 = J_1 \cos\Delta - S_1 \sin\Delta \qquad (36)$$

we can write the first switching equation, Eq. (4), as

$$2(S_1 + S_2) + 2J_1[2\theta x_1 + (1 - \theta^2)y_1](1 + \theta^2)^{-2} - 2S_1[2y_1\theta - x_1(1 - \theta^2)](1 + \theta^2)^{-2} - \theta^{-1}(1 + x_1)(T_1 - T_2) - (2x_1T_2\theta + 2y_1T_2)(1 + \theta^2)^{-1} = 0$$

To the limit

$$4S_1 + 2y_1J + 2x_1S_1 - 2y_1T_1 - r_1U = 0 \qquad (37)$$

By eliminating U between the Eqs. (35) and (37) we have the third relation for singular arc

$$J = [r_1(1 - 3S_1^2 - T_1^2) + (y_1T_1 - S_1)S_1]/(y_1S_1 - 3r_1T_1) \qquad (38)$$

The Eqs. (32, 34, and 38) can now be solved for J, x_1, and y_1, using S_1 and T_1 as parameter. We have, after some long algebraic manipulation

$$y_1 = e \sin v_1 = 2S_1T_1(2 + T_1^2 - 5S_1^2)(3T_1^2 - S_1^2)/(S_1^2 + T_1^2)(2T_1^2 - 3S_1^2 + 1)^2 \qquad (39)$$

$$1 + x_1 = 1 + e \cos v_1 = (1 - 3S_1^2)(3T_1^2 - S_1^2)^2/(S_1^2 + T_1^2)(2T_1^2 - 3S_1^2 + 1)^2 \qquad (40)$$

with

$$S_1 = \sin\phi_1,\ T_1 = \cos\phi_1 \cos\psi_1 \qquad (41)$$

We see that $e \sin v_1$ and $e \cos v_1$ are expressed explicitly in terms of the thrusting angles ϕ_1 and ψ_1. From these, we have

$$\tan v_1 = y_1/x_1 \qquad (42)$$

and

$$e^2 = x_1^2 + y_1^2 \qquad (43)$$

For the planar case, $\psi_1 = 0$, $T_1^2 = 1 - S_1^2$, and we have the well known expressions

$$e \sin v_1 = 6S_1(1 - 2S_1^2)(3 - 4S_1^2)(1 - S_1^2)^{1/2}/(3 - 5S_1^2)^2$$
$$e \cos v_1 = -3S_1^2(7 - 21S_1^2 + 16S_1^4)/(3 - 5S_1^2)^2 \qquad (44)$$

With the expressions (39) and (40) we can now extend Lawden's spiral to the three-dimensional case. References 2, 10, and 11 presented enlightening discussions of the Lawden's spiral in the planar case. For the planar case, Lawden's spiral exists for the values of the eccentricity between 0 and 0.9249. At this limiting value the function $F(v)$, as given by Eqs. (14) and (15) vanishes for another value v_3 of the true anomaly. The primer locus has a contact point of order two (Lawden's point) and another contact point of order unity with the unit circle. Exact solution for this limiting case was given in Ref. 6 as

$$v_1 = v_2,\ \phi_1 = \phi_2,\ \psi_1 = \psi_2 = 0,\ n = \cos 2\phi_1 \qquad (45a)$$

$$1764n^6 + 1890n^5 + 207n^4 - 572n^3 - 309n^2 - 60n - 4 = 0 \qquad (45b)$$

$$e^2 = 9(1 - n)(112n^4 + 119n^3 + 47n^2 + 9n + 1)/(5n + 1)^2 \qquad (45c)$$

$$\tan v_1 = 4n(2n + 1)(1 - n^2)^{1/2}/(n - 1)(8n^2 + 5n + 1) \qquad (45d)$$

$$\tan\frac{v_3 - v_1}{2} = \frac{2(2n + 1)(9n^3 + 19n^2 + 7n + 1)(1 - n^2)^{1/2}}{n(n + 1)(48n^3 + 79n^2 + 31n + 4)} \qquad (45e)$$

$$\sin\phi_3 = \frac{[2(1 - n)]^{1/2}(-90n^5 - 51n^4 + 48n^3 + 62n^2 + 26n + 5)}{2(168n^5 + 267n^4 + 176n^3 + 44n^2 - 4n - 3)} \qquad (45f)$$

The solution is obtained by solving a sixth degree equation in n. Like in the planar transfer, Lawden's arc is the limiting case of the Forward-Forward (or Rearward-Rearward) family of optimal arcs. By Eq. (42), for each prescribed eccentricity, the Lawden's point is not fixed but varies as function of the azimuth ψ_1 of the thrust direction. Although for the planar transfer, Lawden's arc ceases to exist for the value of the eccentricity beyond 0.9249, the critical eccentricity appears sooner in noncoplanar transfer. In the most general case where the point I_1 tends to the point I_2 such that $W_1 \neq W_2$ it is shown in Ref. 12 that the singular switching elements can be expressed explicitly in terms of three parameters, namely the optimum thrust angles ϕ_1 and ψ_1 and a directional parameter

$$V = \lim(W_1 - W_2)/\theta \qquad (46)$$

VI. Cuspidal Case

This case occurs when the two points I_2 and N are coincident (Fig. 5) and form a cusp on the primer locus. When the point I_2 tends to the point N, the two tangents to the primer locus, respectively, at I_2 and N, tend to a common tangent to the unit sphere at a point on the circle, intersection of the plane P and the unit sphere. This cuspidal case is considered as a limiting case, since for a planar transfer, at the cuspidal point, the primer locus changes from a two-loop to a one-loop curve. Also, in the planar case, for each given eccentricity, the cuspidal is very close to the point where the elevation angle ϕ of the optimal thrust direction is highest. By writing that the vector $\mathbf{p}_{12}$ is along the S axis, we have the relation which characterizes the cuspidal case

$$S_1 = S_2 \cos\Delta \to J_2 = 0 \qquad (47)$$

Because of this relation, in three-dimensional transfer, all the optimal arcs for which the primer locus has a cusp at the point I_2 form a family depending on three parameters. We shall use the symmetric variables

$$\theta = \tan(v_2 - v_1)/2,\ \alpha = \tan(\phi_2 - \phi_1)/2 \qquad (48)$$

Then, using the relation (47), it can be shown that the Eqs. (11) and (12) take the form

$$A \cos\psi_2 + B \sin\psi_2 + C = 0 \qquad (49)$$

where

$$A = -\alpha(1 + 3\theta^2)(\alpha^2 + \theta^2) \cos\psi_1 + \theta(\alpha^4 + 6\theta^2\alpha^2 + \theta^4) \qquad (50a)$$

$$B = -\alpha(1 + \theta^2)(\alpha^2 + \theta^2) \sin\psi_1 \qquad (50b)$$

$$C = \theta(\alpha^2 - \theta^2)(\alpha^2 + \theta^2) \cos\psi_1 - \alpha(1 + 3\theta^2)(\alpha^2 - \theta^2) \qquad (50c)$$

Hence, if α, θ, and ψ_1 are given, we can calculate the azimuth

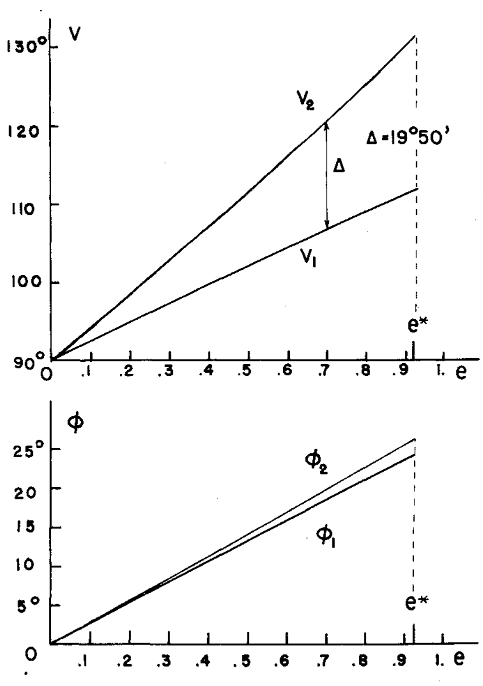

Fig. 8 Optimal switching with cuspidal point.

ψ_2 of the second impulse in closed form by Eq. (49). Next we have

$$\tan(\phi_2 + \phi_1)/2 = \theta^{-2} \tan(\phi_2 - \phi_1)/2 = \alpha\theta^{-2} \quad (51)$$

This relation will give $\phi_2 + \phi_1$ and hence, the elevation angles ϕ_1 and ϕ_2. The components S_i, T_i, W_i of the primer at the switching points are obtained from Eq. (2). Finally, using the identity (33) we can rewrite the switching relations (6) and (7) as a linear system in x_1 and y_1 which, upon solving, will provide the values x_1 and y_1. Therefore, we have the other parameters

$$\tan v_1 = y_1/x_1,\ e^2 = x_1^2 + y_1^2,\ v_2 = \Delta + v_1 \quad (52)$$

All the optimal arcs of the cuspidal family can be obtained that way in closed form. The computation starting from a prescribed eccentricity is more difficult.

The planar case is particularly simple. Since the two parameters $\psi_1 = \psi_2 = 0$ are known, the family of optimal arcs, involving a cuspidal point on the primer locus, depends on one single parameter. Equation (49) is reduced to the quadratic equation in α with θ as parameter

$$\theta\alpha^2 - (1 + 3\theta^2)\alpha + 3\theta^3 = 0 \quad (53)$$

Or, using the parameter

$$m = \cos\Delta \quad (54)$$

we have explicitly for the other optimum elements

$$\tan\phi_1 = \frac{m(1 - m^2)^{1/2}[2(2 - m) + (1 + 2m - 2m^2)^{1/2}]}{(1 + m)(2m^2 - 6m + 5)} \quad (55a)$$

$$\tan\phi_2 = \frac{(1 - m^2)^{1/2}[(2 - m) + 2(1 + 2m - 2m^2)^{1/2}]}{m(1 + m)(4 - 3m)} \quad (55b)$$

$$\theta = [(1 - m)/(1 + m)]^{1/2} \quad (55c)$$

$$\alpha = [(2 - m) - (1 + 2m - 2m^2)^{1/2}](1 - m^2)^{-1/2} \quad (55d)$$

$$e \cos v_2 = -\alpha^3(1 - \theta^4)/\theta(\alpha - \theta)^2(\alpha^2 + \theta^4) \quad (55e)$$

$$e \sin v_2 = (2 + e \cos v_2) \tan\phi_2,\ v_1 = v_2 - \Delta \quad (55f)$$

We see that all the cuspidal optimal arcs, in the planar case, can be obtained explicitly by scanning the whole range of permissible values of the transfer angle Δ. The cuspidal point occurs in a Forward-Forward switching at the second impulse and in a Rearward-Rearward switching at the first impulse. Figure 8 plots the positions of the impulses on the transfer orbit, and the optimum thrust angles for different values of the eccentricity. The graph corresponds to a Forward-Forward switching on the first half of the transfer orbit. The points on the second half of the orbit can be obtained by symmetry with respect to the major axis, and the Rearward-Rearward case is obtained by an inverse operation. The transfer angle Δ varies from zero for circular orbit to a value $\Delta^* = 19°50'24''$. For this value of Δ the primer locus, in the planar case, has three contact points with the unit circle (Fig. 9).

The condition (16) leads to the equation for m

$$81m^6 + 54m^5 - 531m^4 - 180m^3 + 1700m^2 - 1440m + 320 = 0$$

$$m = \cos\Delta^* = 0.94064,\ \Delta^* = 19°50'24'' \quad (56)$$

At this critical value of m

$$\cos v_2^* = 9m(1 - m)/[(10m^2 - 22m + 7) + 2(1 + m)(1 + 2m - 2m^2)^{1/2}]$$

$$\cos v_2^* = -0.66666,\ v_2^* = 131°48'35'',\ v_1^* = 111°58'11'' \quad (57)$$

and

$$e^* = 0.927122,\ \tan\phi_1^* = 0.46369,\ \phi_1^* = 24°52'36'' \quad (58a)$$

$$\tan\phi_2^* = 0.49999,\ \phi_2^* = 26°33'52'' \quad (58b)$$

The third impulse is located at the true anomaly v_3^* such that

$$\tan v_3^* = \tan\Delta^*,\ v_3^* = 180° + \Delta^* = 199°50'24'' \quad (59)$$

It has been shown in Ref. 6 that, when this three-way switching occurs, we have the exact relation

$$\sin\phi_1 \sin(v_3 - v_2) + \sin\phi_2 \sin(v_1 - v_3) + \sin\phi_3 \sin(v_2 - v_1) = 0 \quad (60)$$

Therefore, the optimum angle of the third impulse is

$$\sin\phi_3^* = -\cos v_1^* \sin\phi_2^*,\ \phi_3^* = 170°22' \quad (61)$$

A three-impulse transfer of the type Forward-Forward-Rearward can be constructed this way with the second impulse being infinitesimal at the cuspidal point I_2 so that the primer locus remains unchanged at this point. Beyond this eccentricity limit, cuspidal point no longer exists, at the impulse, for planar transfer because condition (16) is reached before a cusp occurs on the primer, at the point I_2. For a Forward-Forward (or Rearward-Rearward) family of optimal arcs in the plane, the primer locus is always a three-loop or a two-loop curve beyond the critical eccentricity.

VII. Conclusion

The use of the primer vector concept leads to a set of necessary conditions which must be satisfied by all optimal arcs

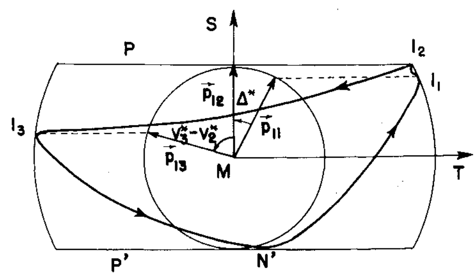

Fig. 9 Cuspidal primer locus at critical eccentricity.

connecting two switching points in a three-dimensional, time-free, impulsive transfer. The set of switching equations are sufficient in number for a computation and cataloguing of all optimal arcs. These arcs can be classified into two families. The family of Forward-Forward (or Rearward-Rearward) switching tends to the Lawden's singular arc when the transfer angle tends to zero. The family of Forward-Rearward switching tends to the symmetric arc. These limiting cases are considered in this paper. It is shown that all the elements of the symmetric arc can be expressed in terms of two parameters and the elements of singular arc in terms of three parameters. For the family of all symmetric arcs, the parameters selected are the eccentricity of the coasting arc and the azimuth of the initial thrust direction. For the family of all singular arcs, the parameters used are the azimuth and the elevation angle of the thrust direction and a directional parameter.

Another limiting case is considered in this paper, namely the case in which the primer locus has a cusp at a switching point. For a planar transfer this case is a limiting case in the sense that the primer locus changes from a two-loop curve to a one-loop curve. The cuspidal arc belongs to the family of Forward-Forward (or Rearward-Rearward) switching. For three-dimensional transfer, the family of all cuspidal arcs depends on three parameters, namely the transfer angle, the difference between the optimum elevation angles and the azimuth of the initial thrust direction.

The study presented in this paper would greatly facilitate the task of computation and tabulation of all optimal arcs in the three-dimensional, time-free, optimum transfer between orbits.

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